

# Integral operators on interpolation sets in $\mathbb{C}^n$

by  
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## Abstract

In this paper we study interpolation sets  $K$  in  $\mathbb{C}^n$ . The paper is divided into two parts. In the first part we prove a general existence theorem for an integral-operator producing  $A(\overline{\Omega})$  extensions of any  $f$  in  $C(K)$ .

In part two we consider a paper by A. Nagel (6) "Smooth zero sets and interpolation sets for some algebras of holomorphic functions on strictly pseudoconvex domains". In his paper, Nagel constructs an explicit extension operator. We have been given the impression, however, that parts of Nagel's proofs are hard to follow. The proof presented here is essentially the same as Nagel's proof, but we believe we have simplified some obscure points. Our global theory is quite different, and is in our formulation of the problem, an almost immediate consequence of the local results.

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## Part 1

**A general existence theorem for an integral operator on peak-interpolation sets in  $\mathbb{C}^n$ .**

In "Function Theory in the Unit Ball of  $\mathbb{C}^n$ " (1), Rudin poses the following question: (19.3.13 p. 416)

If  $K \subseteq \partial\Omega$  is an interpolation set for  $A(\overline{\Omega})$ , is there an integral operator that produces  $A(\overline{\Omega})$  extensions of any  $f$  in  $C(K)$ ?

It is the purpose of this section to prove that such operators always exist. The deep part of this result is a theorem from Davie (2).

## Theorem (Davie)

Let  $X$  be a compact metric space, and let  $A$  be a closed subalgebra of  $C(X)$  separating points and containing the constant functions. If  $K$  is a peak-interpolation set for  $A$ , then there exists a continuous linear extension operator  $T : C(K) \rightarrow A$ .

If we assume that  $\Omega$  is compact, the theorem obviously applies to our situation. Given this extension operator, we can get an integral extension from a construction developed by Gleason (3). Indeed, an  $L^1$  version of our theorem could be proved directly from the results in this paper. (i.e. thms 2.14 and 3.11 should do the job). However, Bungart (4) has pointed out an ingenious trick that makes  $L^\infty$ -kernels possible.

My proof is a modification of the proofs in (3) and (4).

### Theorem

Let  $\Omega \subset \mathbb{C}^n$  be a domain and let  $K \subset \partial\Omega$  be a compact peak-interpolation set for  $A(\overline{\Omega})$ . Then there exist a continuous linear operator  $T : C(K) \rightarrow A(\overline{\Omega})$ , a measure  $\sigma$  on  $K$  and a  $\mu x \sigma$ -measurable\* function  $\phi : \Omega \times K \rightarrow \mathbb{C}$  s.t.

- (i)  $Tf(z) = f(z)$  on  $K$
- (ii)  $Tf(z) = \int_K f(\xi) \phi(z, \xi) d\sigma(\xi)$  on  $\Omega$
- (iii) For  $z \in \Omega$  fixed,  $\xi \mapsto \phi(z, \xi)$  is in  $L^\infty(\sigma)$
- (iv) For  $\xi \in K$  fixed,  $z \mapsto \phi(z, \xi)$  is holomorphic.

\*  $\mu$  is Lebesgue-measure on  $\mathbb{C}^n$ .

### Proof

For  $z_0$  in  $\Omega$  let  $D(z_0, r_0)$  denote the maximal uniform polydisc about  $z_0$  contained in  $\Omega$ . For each multi-index  $j$ , we consider the linear functionals  $\delta_{z_0, j}$  in  $C^*(\overline{\Omega})$  defined by

$$\langle F, \delta_{z_0, j} \rangle = \frac{1}{(2\pi i)^n} \int_{\partial D(z_0, r_0)} \frac{F(\xi) d\xi}{(\xi - z_0)^{(j)+I}}$$

The usual estimates on these integrals give

$$(*) \quad \|\delta_{z_0, j}\| \leq r_0^{-|j|}$$

Since  $A(\overline{\Omega})$  is an algebra of holomorphic functions, we locally have

$$\begin{aligned} Tf(z) &= \sum_{(j)} \langle Tf, \delta_{z_0, j} \rangle (z - z_0)^{(j)} \\ &= \sum_{(j)} \langle f, T^* \delta_{z_0, j} \rangle (z - z_0)^{(j)} \\ &= \sum_{(j)} \int_K f d\mu_{z_0, j} (z - z_0)^{(j)} \end{aligned} \quad (**)$$

(We have identified  $T^* \delta_{z_0, j}$  in  $C^*(K)$  with a measure  $\mu_{z_0, j}$ ).

If  $z_0$  is allowed to run through a countable dense subset  $\{z_m\}_{m=1}^\infty$  of  $\Omega$ , this gives us a global representation of  $T$  by a countable number of measures. The idea is now to use the estimate (\*) to sum all these measures together to a single measure  $\sigma$ , and then use Radon-Nikodym derivatives in (\*\*) to get a local kernel. More precisely we do the following: Let

$$\nu_j^m = \mu_{z_m, j} \cdot (r_m/2)^{|j|}$$

By (\*)

$$\|\nu_j^m\| = \|\mu_{z_m, j}\| (r_m/2)^{|j|} = \|T^* \delta_{z_m, j}\| (r_m/2)^{|j|} \leq \|T\| \cdot 2^{-|j|}$$

So

$$C_m = \sum_{(j)} \|\nu_j^m\| < \infty$$

We define the measure  $\sigma$  by

$$\sigma = \sum_m \sum_{(j)} 2^{-m} C_m^{-1} |\nu_j^m|$$

We obviously have  $|\nu_j^m| \leq 2^m C_m \sigma$  so by the Radon-Nikodym theorem there exist  $\sigma$ -measurable functions  $f_j^m$  s.t.

$$\int_K f d\nu_j^m = \int_K f(\xi) f_j^m(\xi) d\sigma(\xi)$$

and we can assume that  $|f_j^m(\xi)| \leq 2^m C_m$  everywhere on  $K$ . When we use these functions in (\*\*)

$$\begin{aligned} Tf(z) &= \sum_{(j)} \int_K f d\mu_{z_m, j}(z - z_m)^{(j)} \\ &= \sum_{(j)} \int_K f d\nu_j^m \left(\frac{2}{r_m}\right)^{|j|} (z - z_m)^{(j)} \\ &= \sum_{(j)} \int_K f(\xi) f_j^m(\xi) (2/r_m)^{|j|} (z - z_m)^{(j)} d\sigma(\xi) \end{aligned}$$

The maximality of  $r_m$  allows us to cover  $\Omega$  by the polydiscs  $D_m = D(z_m, r_m/3)$ . Now it is easy to see that on  $D_m$  there is a function  $\phi_m(z, \xi)$  s.t.

$$\sum_{(j)}^{|j| \leq N} f_j^m(\xi) (2/r_m)^{|j|} (z - z_m)^{(j)} \rightarrow \phi_m(z, \xi) \text{ uniformly in } \xi$$

This makes it possible to interchange summation and integration, and then  $\phi_m(z, \xi)$  clearly is a local kernel on  $D_m$ . Since  $|f_j^m(\xi)| \leq 2^m C_m$  independent of  $j$ , this kernel satisfies the conditions (iii) and (iv) in the theorem. To complete the proof, it suffices to find local kernels that coincide on the intersection of their domains. For each  $i, j$  let  $\{z_{ijk}\}_{k=1}^\infty$  be dense in  $D_i \cap D_j$ .

Since for each  $ijk$ , and all  $f \in C(k)$

$$\int_K f(\xi) \phi(z_{ijk}, \xi) d\sigma(\xi) = \int_K f(\xi) \phi_j(z_{ijk}, \xi) d\sigma(\xi)$$

the mappings  $\xi \mapsto \phi_i(z_{ijk}, \xi)$  and  $\xi \mapsto \phi_j(z_{ijk}, \xi)$  are equal except on a set  $E_{ijk}$  of  $\sigma$ -measure zero. We redefine the local kernels by setting  $\phi_i(z, \xi) = 0$  if  $\xi \in \bigcup_{i,j,k} E_{ijk}$ . Since

the kernels are continuous in the first argument, we will actually get  $\phi_i(z, \xi) = \phi_j(z, \xi)$  on  $D_i \cap D_j) \times K$  and this proves the theorem. ■

## Part II

### Theorem (Nagel)

Let  $\Omega \subset \subset \mathbb{C}^n$  be strictly pseudoconvex with  $C^3$  boundary. Let  $M$  be a proper complex tangential submanifold of  $\partial\Omega$  of class  $C^3$ . If  $K \subset M$  is compact, there exists  $U$  open in  $M$  s.t.  $K \subset U$  and an integral extension operator  $T : C_c(U) \rightarrow A(\overline{\Omega})$ .

By a Tietze extension from  $C(K)$  to  $C_c(U)$  it follows that  $K$  is an interpolation set for  $A(\overline{\Omega})$ . To prove this theorem we will systematically exploit the two facts below:

A) If  $\Omega \subset \subset \mathbb{C}^n$  is a strictly pseudoconvex domain with  $\rho$  a  $C^2$  defining function s.t.  $L_{\rho(p)}(a, a) \geq C|a|^2$ , then there exist constants  $C^1, \epsilon > 0$  s.t.  $\operatorname{Re} F(z, w) \geq C^1|z - w|^2$  if  $(z, w) \in \Omega \times \partial\Omega, |z - w| < \epsilon$  where  $F$  is the Levi polynomial.

B) Let  $[c_{kl}]$  be a strictly positive definite real symmetric  $n \times n$ -matrix with inverse  $[\gamma_{kl}]$  If  $\operatorname{Re} z > 0, p > \frac{n}{2}$  and  $a_1 a_2, \dots, a_n$  are any real numbers, then

$$\begin{aligned} & \int_{\mathbb{R}^n} (z + 2i \sum_{k=1}^n a_k t_k + \sum_{k,l=1}^n c_{kl} t_k t_l)^{-p} dt \\ &= C(n, p) \det[c_{kl}]^{-\frac{1}{2}} (z + \sum_{k,l=1}^n \gamma_{kl} a_k a_l)^{\frac{n}{2}-p} \end{aligned}$$

A is well known. To prove B consider the case  $n = 1$   $C_{11} = 1$   $a > 0$  and note that  $\operatorname{Re}(w^2 + 2iaw + z) = (\operatorname{Re} w)^2 - \operatorname{Im} w \cdot \operatorname{Im}(w + 2ia) + \operatorname{Re} z$ . If  $-2a \leq \operatorname{Im} w \leq 0$ , then  $\operatorname{Re}(w^2 + 2iaw + z) > 0$ . By contour integration this enables us to write

$$\begin{aligned} & \int_{-\infty}^{\infty} (z + 2iat + t^2)^{-p} dt = \int_{-\infty}^{\infty} [z + a^2 + (t + ai)^2]^{-p} dt \\ &= \int_{-\infty}^{\infty} (z + a^2 + t^2)^{-p} dt = (z + a^2)^{\frac{1}{2}-p} \int_{-\infty}^{\infty} (1 + t^2)^{-p} dt \\ &= C(n, p) (z + a^2)^{\frac{1}{2}-p} \text{ and similarly if } a \leq 0. \end{aligned}$$

In the general case you diagonalize the quadratic form, and apply the 1-dimensional result to each variable. ■

We first look at an example of how to use A and B to reach the conclusion.

### Example

We let  $\Omega \subset \mathbb{C}^{n+1}$  have a defining function

$$\rho(z) = 4 \sum_{j=1}^n (\operatorname{Im} z_j)^2 + 2 \operatorname{Re}(z_{n+1}).$$

We consider the complex-tangential submanifold  $M \subset \partial\Omega$  given by

$$M = \{z \in \mathbb{C}^{n+1} | z = (t_1, t_2, \dots, t_n, 0) \quad t \in \mathbb{R}^n\}$$

A short computation shows that the Levi-polynomial  $F$  is given by

$$F(z, t) = \sum_{j=1}^n (t_j - z_j)^2 - z_{n+1} \text{ if } z \in \mathbb{C}^{n+1}, t \in M.$$

We then note that  $F$  has a trivial 2.order Taylor expansion in  $t$ , i.e.

$$F(z, t) = F(z, s) + \sum_{j=1}^n 2(s_j - z_j)(t_j - s_j) + \sum_{j=1}^n (t_j - s_j)^2$$

For each  $z$  we can choose  $s \in M$  s.t  $\operatorname{Re}(s_j - z_j) = 0 \quad j = 1, 2, \dots, n$ . We want to integrate  $F$  by the integration formula B. For this we need  $\operatorname{Re} F(z, s) > 0$ . This is nearly what we get from A. We are actually cheating a little, the Levi-form in this case is only semi-definite. It is not hard, however, to see that  $\operatorname{Re} F(z, s) \geq 0$  when  $z \in \overline{\Omega} \setminus M$ , and by a minor continuity argument you realize that the integration formula B still applies i.e.

$$\begin{aligned} T_p 1(z) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \frac{dt}{F(z, t)^p} = \int_{\mathbb{R}^n} \frac{dt}{(F(z, s) + 2i \sum_j i^{-1}(s_j - z_j)t_j + \sum_j t_j^2)^p} \\ &= C(n, p) \{F(z, s) + \sum_j i^{-2}(s_j - z_j)^2\}^{n/2-p} \\ &= C(n, p)(-z_{n+1})^{n/2-p} \end{aligned}$$

By definition of  $\Omega$ ,  $\operatorname{Re} z_{n+1} \leq 0$  if  $z \in \overline{\Omega}$ . In case  $\operatorname{Re} z_{n+1} = 0$  then  $\operatorname{Im} z_j = 0$  for  $j = 1, 2, \dots, n$  and we see that  $\operatorname{Im} z_{n+1} \neq 0$  if we avoid  $M$ . We conclude that  $T_p 1(z)$  is well defined and never zero on  $\overline{\Omega} \setminus M$ . For  $f \in C_c(M)$  we define

$$\begin{aligned} T_p f(z) &= \int_{\mathbb{R}^n} \frac{f(t) dt}{F(z, t)^p} \text{ and} \\ T f(z) &= \begin{cases} T_p f(z) / T_p 1(z) & z \in \overline{\Omega} \setminus M \\ f(z) & z \in M \end{cases} \end{aligned}$$

then gives the required extension operator.

If we let  $p = \frac{n+1}{2}$  we have a striking similarity with the Poisson kernel in a half-space in  $\mathbb{R}^{n+1}$ . In this case

$$Tf(z) = \int_{\mathbb{R}^n} \frac{C(n)(-z_{n+1})^{\frac{1}{2}} f(t) dt}{\left(\sum_{j=1}^n (t_j - z_j)^2 - z_{n+1}\right)^{n/2 + \frac{1}{2}}}$$

$$Pf(x, y) = \int_{\mathbb{R}^n} \frac{C(n)y f(t) dt}{\left(\sum_{j=1}^n (t_j - x_j)^2 + y^2\right)^{n/2 + \frac{1}{2}}}$$

### General case-an outline

We prove the general case in exactly the same way. It will, however, be necessary with some technical modifications. First of all we cannot expect to get  $\operatorname{Re} F(z, w) > 0$  everywhere, so we introduce a cut-off function  $\psi \in C_c^\infty(\mathbb{C}^n)$  s.t.

$$\psi(\xi) = 1 \text{ when } |\xi| < \frac{\epsilon}{4} \quad \psi(\xi) = 0 \text{ when } |\xi| > \frac{\epsilon}{2}$$

where  $\epsilon$  is the constant from A i.e.  $\operatorname{Re} F(z, w) \geq C|z - w|^2$  if  $|z - w| < \epsilon$ . We let  $\mu$  denote Lebesgue-measure on  $M$  and define

$$T_p f(z) = \int_{\mu} \frac{f(w) \psi(z - w) d\mu(w)}{F(z, w)^p}$$

The proof then goes as follows.

- i) We show that the Levi-polynomial  $F$  has a second order Taylor expansion s.t. in terms of the Levi-form  $L$

$$F(z, \phi(x)) = F(z, \phi(y)) + L_{\rho(\phi(y))}(\phi(y) - z, \phi_*) + \frac{1}{2} L_{\rho(\phi(y))}(\phi_*, \phi_*) + \text{Error term}$$

- ii) For each  $z$  we pick  $y$  s.t.  $L_{\rho(\phi(y))}(\phi(y) - z, \phi_*)$  is pure imaginary.
- iii) Since the Levi-form is strictly positive definite it is easy to see that the second order terms form a strictly positive definite real symmetric  $m \times m$ -matrix.
- iv) We note that ii) and iii) makes it possible to integrate the second order expansion by the formula B. Let  $H$  denote this integral.
- v) By long and tedious verification we show that  $H$  gives the “right” asymptotical behaviour for  $T_p f$  as we approach  $M$ .
- vi) We find a nice compact  $\tilde{K}$  including the original compact in the theorem, and define

$$T_p \tilde{K}(z) = \int_{\tilde{K}} \frac{\psi(z - w) d\mu(w)}{F(z, w)^p}$$

- vii) We correct the kernel by a uniformly bounded solution of  $\bar{\partial}$ , and add a possibly large real constant to the kernel to assure that  $\operatorname{Re} T_p \tilde{K}(z) > 0$  everywhere.

viii)  $Tf(z) = T_p f(z)/T_p \tilde{K}(z)$  then gives the extensions.

### General case-details

We will now fill in the details to complete the proof. We start with a lemma.

#### Lemma 1

$$\begin{aligned} F(z, w) - F(z, b) &= L_{\rho(b)}(b - z, w - b) + \frac{1}{2} L_{\rho(b)}(w - b, w - b) \\ &\quad + \sum_i (w_i - b_i) \rho_i(b) \\ &\quad + \frac{1}{2} \sum_{i,j} (w_i - b_i)(w_j - b_j) \rho_{ij}(b) \\ &\quad + \frac{1}{2} \sum_{i,j} (w_i - b_i)(\bar{w}_j - \bar{b}_j) \rho_{i\bar{j}}(b) \\ &\quad + \{|z - b|^2 + |w - b|^2\} 0\{|w - b|\} \end{aligned}$$

#### Proof

We want to use a second order Taylor expansion in  $w$ . To do this you really need  $\rho \in C^4$ . It is fairly straightforward to see that you also have this estimate for  $\rho \in C^3$ . The details, however, are tedious, so we only prove the case  $\rho \in C^4$ . Up to significant terms we have

$$\begin{aligned} \frac{\partial F}{\partial w_i} &= \rho_i(w) & \frac{\partial F}{\partial \bar{w}_j} &= \sum_i (w_i - z_i) \rho_{i\bar{j}}(w) \\ \frac{\partial^2 F}{\partial w_i \partial w_j} &= \rho_{ij}(w) & \frac{\partial^2 F}{\partial w_i \partial \bar{w}_j} &= \rho_{i\bar{j}}(w) & \frac{\partial^2 F}{\partial \bar{w}_i \partial \bar{w}_j} &= 0 \end{aligned}$$

Except from a convenient rearrangement of terms, the estimate is just Taylors formula. ■

#### Lemma 2

If  $\phi : B \rightarrow \mathbb{R}^n$  is any  $C^3$ -function  $w = \phi(x)$   $b = \phi(y)$  then

$$\begin{aligned} &\sum_i (w_i - b_i) \rho_i(b) \\ &\quad + \frac{1}{2} \sum_{i,j} (w_i - b_i)(w_j - b_j) \rho_{ij}(b) \\ &\quad + \frac{1}{2} \sum_{i,j} (w_i - b_i)(\bar{w}_j - \bar{b}_j) \rho_{i\bar{j}}(b) \\ &= \sum_k \left\{ \sum_i \frac{\partial \phi_i}{\partial x_k}(y) \rho_i(b) \right\} (x_k - y_k) \\ &\quad + \frac{1}{2} \sum_{k,l} \left\{ \frac{\partial}{\partial y_l} \left\{ \sum_i \frac{\partial \phi_i}{\partial x_k}(y) \rho_i(b) \right\} \right\} (x_k - y_k)(x_l - y_l) \\ &\quad + |x - y|^2 0\{|x - y|\} \end{aligned}$$

**Proof**

Estimate  $\phi$  to 2.order. The lemma then follows immediately from the chain rule. ■

If  $\phi$  happens to point in the complex direction, lemma 1 and 2 gives the desired Taylor expansion.

**Taylor expansion**

If  $M$  points in the complex direction and  $\phi : B \rightarrow M$  is a  $C^3$  parametrization, then

$$F(z, \phi(x)) = F(z, \phi(y)) + L_{\rho(\phi(y))}(\phi(y) - z, \phi_*) + \frac{1}{2} L_{\rho(\phi(y))}(\phi_*, \phi_*) \\ + \{|z - \phi(y)|^2 + |x - y|^2\} O\{|x - y|\}$$

where  $\phi_* = \sum_{k=1}^m \frac{\partial \phi}{\partial x_k}(y)(x_k - y_k)$

i) ■

**Local considerations**

We are now going to study the local behaviour of  $F$  as we approach a fixed point  $u \in M$ . We will assume that  $\phi : B \rightarrow M$  is a parametrization s.t.  $\phi(0) = u$ .

For  $\xi \in M$  we define

$$T_\xi M^\perp = \{v \in \mathbb{C}^n | \operatorname{Re} L_{\rho(\xi)}(v, w) = 0 \text{ for all } w \in T_\xi M\}$$

$TM^\perp$  is then a smooth vectorbundle over  $M$ . We use this to prove the following proposition.

**Proposition 1**

Let  $u \in M$ . Then there exist a ball  $V$  about  $u$  and a smooth mapping  $\theta : V \rightarrow M$  s.t.

- a)  $z - \theta(z) \in T_{\theta(z)} M^\perp$  for all  $z \in V$ .
- b)  $z = \theta(z)$  when  $z \in V \cap M$ .

**Proof**

Since  $TM^\perp$  is a smooth vector bundle over  $M$ , there exist a neighbourhood  $W$  about  $u$  in  $M$  and  $\phi$  a smooth trivialization over  $W$ . Define

$$\gamma : W \times \mathbb{R}^{2n-m} \rightarrow \mathbb{C}^n \text{ by} \\ \gamma(\rho, v) = (\pi_1 + \pi_2)\phi(p, v)$$

Then  $\gamma$  is a local diffeomorphism and  $\gamma(p, 0) = p$ . For  $V$  a sufficiently small ball about  $u$ , we define  $\gamma^{-1} : V \rightarrow W \times \mathbb{R}^{2n-m}$  and put  $\theta = \pi_1 \circ \gamma^{-1}$  ■

**Choice of expansion point**

From here on  $V_k$  will always denote a ball about  $u$ . Whenever we choose  $V_k$ , we will tacitly assume that we choose a ball strictly smaller than all the ones chosen before. We



start with the ball  $V_0$  from proposition 1, and choose  $V_1$ . We also choose  $\tilde{B} \subset\subset B$  s.t.  $\phi(\tilde{B}) \subset V_1$  and choose  $V_2$  s.t.  $\theta(V_2) \subset \phi(\tilde{B})$ . For  $z \in V_2$  we choose the expansion point  $y$  by  $y = \phi^{-1}(\theta(z)) \in \tilde{B}$ . Proposition 1 now says that  $L_{\rho(\phi(y))}(\phi(y) - z, \phi_*)$  is pure imaginary.

ii) ■

### Distance estimate

For  $z \in V_2$  there exist  $C < \infty$  s.t.

$$\text{dist}(z, \phi(\tilde{B})) \leq |z - \theta(z)| \leq C \text{dist}(z, \phi(\tilde{B})).$$

### Proof

Let  $z \in \bar{V}_1$ . Since  $\theta$  is smooth,  $\bar{V}_1$  is compact and  $V_2 \subset \bar{V}_1$ ,  $|\theta(z) - \theta(z_0)| \leq C|z - z_0|$ . If  $z_0 \in \bar{V}_1 \cap M$ ,  $\theta(z_0) = z_0$  and

$$|\theta(z) - z| \leq |\theta(z) - \theta(z_0)| + |z - z_0| \leq (C + 1)|z - z_0|$$

This proves the second inequality. The first is trivial. ■

### Quadratic terms

Define matrices  $[C_{kl}(y)]$  by

$$C_{kl}(y) = \text{Re} L_{\rho(\phi(y))} \left( \frac{\partial \phi}{\partial x_k}(y), \frac{\partial \phi}{\partial x_l}(y) \right) \quad k, l = 1, 2, \dots, m$$

Since

$$\begin{aligned} \sum_{k,l} C_{kl}(y)(x_k - y_k)(x_l - y_l) &= \text{Re} L_{\rho(\phi(y))}(\phi_*, \phi_*) \\ &= L_{\rho(\phi(y))}(\phi_*, \phi_*) \end{aligned}$$

these matrices are all real symmetric  $m \times m$  matrices and their eigenvalues are uniformly bounded away from 0 and  $\infty$  on  $\tilde{B}$  by constants  $\delta, D$   $0 < \delta < D < \infty$ . We let  $[\gamma_{kl}(y)]$  denote the inverse matrices.

iii) ■

To study the asymptotical behaviour of  $Tpf$  we need some technical tools. First some notation.

### Notation

For  $z \in V_2, y = y(z)$  and  $x \in \mathbb{R}^m$  we define

$$a_k(z, y) = -i L_{\rho(\phi(y))}(\phi(y) - z, \frac{\partial \phi}{\partial x_k}(y)) \quad k = 1, 2, \dots, m$$

$$G(z, y, x) = F(z, \phi(y)) + i \sum_k a_k(z, y)(x_k - y_k) + \frac{1}{2} \sum_{k,l} C_{kl}(y)(x_k - y_k)(x_l - y_l)$$

$$Q(z, y) = 2F(z, \phi(y)) + \sum_{k,l} \gamma_{kl}(y) a_k(z, y) a_l(z, y)$$

(i.e)  $G$  is the second order approximation of  $F$  and  $Q$  is the resulting expression when we apply the integration formula B to  $2G$ .

### Technical estimates

Let  $\epsilon$  be the constant from A and choose  $V_3$  s.t.  $|z - \phi(y)| < \epsilon$  for all  $z \in V_3$ . This is possible by the distance estimate above. For  $z \in V_3 \cap \bar{\Omega}$   $y = y(z)$  and  $x \in \mathbb{R}^m$  we have

$$1a) \quad a_k(z, y)^2 \leq C \operatorname{Re} F(z, \phi(y))$$

$$1b) \quad \text{There exists } 0 < s_0 < 1 \text{ s.t.}$$

$$\operatorname{Re} F(z, \phi(y)) - s_0 \|a\|^2 \geq \frac{1}{2} \operatorname{Re} F(z, \phi(y))$$

$$1c) \quad \operatorname{Re} G(z, y, x) \geq C\{|z - \phi(y)|^2 + |x - y|^2\}$$

$$1d) \quad |F(z, \phi(y)) - G(z, y, x)| \leq C|x - y|\{|z - \phi(y)|^2 + |x - y|^2\}$$

$$1e) \quad |Q(z, y)| \leq C|F(z, \phi(y))| \leq C \operatorname{dist}(z, \phi(\tilde{B}))$$

$$1f) \quad \operatorname{Re} Q(z, y) \geq C \operatorname{dist}^2(z, \phi(\tilde{B}))$$

$$1g) \quad \text{There exist } \alpha_0 > 0 \text{ s.t.}$$

$$|F(z, \phi(y)) - G(z, y, x)| \leq \frac{1}{2} |G(z, y, x)|$$

$$\text{when } |z - u| < \alpha_0 \quad |x| < \alpha_0$$

$$1h) \quad \text{If } |z - u| < \alpha_0 \quad |x| < \alpha_0 \text{ then}$$

$$\left| \frac{1}{F^p} - \frac{1}{G^p} \right| \leq C \frac{|x - y|^3 + |x - y||z - \phi(y)|^2}{|G|^{p+1}}$$

$$1i) \quad |G(z, y, x + y)| \geq C\{|F(z, \phi(y))| + |x|^2\}$$

### Proof

a) Is obvious from the definition of A and the choice of  $V_3$

b) Follows from a)  $\|a\|^2 = \sum_{k=1}^m a_k^2(z, y)$ .

c)

$$\begin{aligned} \operatorname{Re} G(z, y, x) &= \operatorname{Re} F(z, \phi(y)) + \frac{1}{2} \sum_{k,l} C_{kl}(y)(x_k - y_k)(x_l - y_l) \\ &\geq C|z - \phi(y)|^2 + \frac{\delta}{2}|x - y|^2 \end{aligned}$$

d) Is just a reformulation of the error term in the Taylor approximation.

e) By definition of the Levi-polynomial we have

$$|F(z, \phi(y))| \leq C|z - \phi(y)| \leq C \text{dist}(z, \phi(\tilde{B}))$$

$$|Q(z, y)| \leq 2|F(z, \phi(y))| + \frac{1}{\delta} \sum a_k(z, y)^2 \leq C|F(z, \phi(y))|,$$

by a) and the fact that  $\frac{1}{\delta}$  is an upper bound of the eigenvalues of  $[\gamma_{kl}(y)]$ .

f) Is trivial

g) By d)

$$|F(z, \phi(y)) - G(z, y, x)| \leq C|x - y|\{|z - \phi(y)|^2 + |x - y|^2\}$$

$$\text{By c) } \leq C|x - y| \text{Re}G(z, y, x)$$

$$\leq C\{|x| + |y|\}|G(z, y, x)|$$

Since the Jacobian of  $\phi$  is of maximal rank

$$|y| = |y - 0| \leq C|\phi(y) - u| \leq C\{|\phi(y) - z| + |z - u|\}$$

$$\leq C\{\text{dist}(z, \phi(\tilde{B})) + |z - u|\} \leq C|z - u|$$

and g) follows.

h)  $|\frac{1}{F^p} - \frac{1}{G^p}| = |\frac{F-G}{(sF+(1-s)G)^{p+1}}|$  for some  $s$  s.t.  $|s| \leq 1$ . h) then follows from d) and g).

i)

$$|G(z, y, x + y)| = |F(z, \phi(y)) + i \langle a, x \rangle + \frac{1}{2} \sum C_{kl}(y)x_k x_l|$$

$$\geq C\{\text{Re}F(z, \phi(y)) + \frac{\delta}{2}\|x\|^2 + |\text{Im}F(z, \phi(y)) + \langle a, x \rangle|\}$$

$$\geq C\{\text{Re}F(z, \phi(y)) + \|x\|^2 + s_0\{|\text{Im}F(z, \phi(y))| - \|a\|^2 - \|x\|^2\}\}$$

$$= C\{\text{Re}F(z, \phi(y)) - s_0\|a\|^2 + s_0|\text{Im}F(z, \phi(y))| + (1 - s_0)\|x\|^2\}$$

$$\text{By b) } \geq C\{\text{Re}F(z, \phi(y)) + |x|^2\}$$

### Integration lemma I

If  $\beta < \alpha_0$  above  $|y| < \beta$  and  $\frac{m}{2} < p < \frac{m}{2} + \frac{1}{2}$

$$\int_{|x| < \beta} |\frac{1}{F^p} - \frac{1}{G^p}| dx \leq C|Q(z, y)|^{\frac{m}{2}-p} \text{dist}^{\frac{1}{2}}(z, \phi(\tilde{B})) + C$$

**Proof**

By 1h)

$$\int_{|x|<\beta} \left| \frac{1}{F^p} - \frac{1}{G^p} \right| dx \leq C \int_{|x|<\beta} \frac{|x-y|^3 + |x-y||z-\phi(y)|^2}{|G(z,y,x)|^{p+1}} dx$$

$$\leq C \int_{|x|<2\beta} \frac{|x|^3 dx}{|G(z,y,x+y)|^{p+1}} + C \int_{\mathbb{R}^m} \frac{|x||z-\phi(y)|^2 dx}{|G(z,y,x+y)|^{p+1}}$$

By 1i)

$$\leq C \int_{|x|<2\beta} \frac{|x|^3 dx}{\{|F(z,\phi(y))| + |x|^2\}^{p+1}}$$

$$+ C \int_{\mathbb{R}^m} \frac{|x||z-\phi(y)|^2 dx}{\{|F(z,\phi(y))| + |x|^2\}^{p+1}}$$

$$\leq C \int_0^{2\beta} \frac{r^{m+2} dr}{\{|F(z,\phi(y))| + r^2\}^{p+1}}$$

$$+ C \int_0^\infty \frac{r^m |z-\phi(y)|^2 dr}{\{|F(z,\phi(y))| + r^2\}^{p+1}}$$

$$\leq C \int_0^{2\beta} \{|F(z,\phi(y))| + r^2\}^{m/2-p} dr$$

$$+ C \int_0^\infty \frac{|z-\phi(y)|^2 dr}{\{|F(z,\phi(y))| + r^2\}^{p+1-\frac{m}{2}}}$$

$$\leq C \int_0^{2\beta} \{|F(z,\phi(y))|^{\frac{1}{2}} + r\}^{m-2p} dr$$

$$+ C \int_0^\infty \frac{|z-\phi(y)|^2 dr}{\{|F(z,\phi(y))|^{\frac{1}{2}} + r\}^{2p+2-m}}$$

$$\leq C|F(z,\phi(y))|^{\frac{m}{2}-p+\frac{1}{2}} + C + C|z-\phi(y)|^2|F(z,\phi(y))|^{\frac{m}{2}-p-\frac{1}{2}}$$

$$= C|F(z,\phi(y))|^{\frac{m}{2}-p}\{|F(z,\phi(y))|^{\frac{1}{2}} + |z-\phi(y)|^2|F(z,\phi(y))|^{-\frac{1}{2}}\} + C$$

By 1e) and the distance estimate

$$\leq C|Q(z,y)|^{\frac{m}{2}-p}\{dist(z,\phi(\tilde{B}))^{\frac{1}{2}} + dist^2(z,\phi(\tilde{B}))|F(z,\phi(y))|^{-\frac{1}{2}}\} + C$$

Now  $|F(z,\phi(y))| \geq ReF(z,\phi(y)) \geq C|z-\phi(y)|^2 \geq Cdist^2(z,\phi(\tilde{B}))$  so

$$\leq C|Q(z,y)|^{\frac{m}{2}-p}dist^{\frac{1}{2}}(z,\phi(\tilde{B})) + C.$$

### Integration lemma II

If  $f$  is continuous in  $\overline{D}(0, \beta)$ ,  $\beta < \alpha_0$  above  $|y| < \beta$  and  $p > \frac{m}{2}$  then

$$\int_{|x| < \beta} \frac{|f(x) - f(y)| dx}{|G(z, y, x)|^p} \leq |Q(z, y)|^{\frac{m}{2} - p} \cdot o(1) \text{ as } \text{dist}(z, \phi(\tilde{B})) \rightarrow 0$$

### Proof

It is enough to prove that for each  $\gamma > 0$  there exist  $d(\gamma) > 0$  s.t.

$$\int_{|x| < \beta} \frac{|f(x) - f(y)| dx}{|G(z, y, x)|^p} \leq C\gamma |Q(z, y)|^{\frac{m}{2} - p} \text{ when } \text{dist}(z, \phi(\tilde{B})) < d(\gamma).$$

where  $C$  is some constant independent of  $\gamma$ . Since  $f$  is uniformly continuous there exist  $\delta(\gamma) > 0$  s.t.  $|f(x) - f(y)| < \gamma$  when  $|x - y| < \delta(\gamma)$ . This gives

$$\int_{|x| < \beta} \frac{|f(x) - f(y)| dx}{|G(z, y, x)|^p} \leq \gamma \int_{|x-y| < \delta(\gamma)} \frac{dx}{|G(z, y, x)|^p} + 2\|f\| \int_{\substack{|x-y| \geq \delta(\gamma) \\ |x| < \beta}} \frac{dx}{|G(z, y, x)|^p}$$

The same estimates as the ones we used in the first integration lemma gives

$$\int_{|x-y| < \delta(\gamma)} \frac{dx}{|G(z, y, x)|^p} \leq C|Q(z, y)|^{\frac{m}{2} - p} + C(\gamma)$$

The second term is bounded by some constant  $C(\gamma)$ , and this proves the lemma since  $|Q(z, y)|^{\frac{m}{2} - p} \rightarrow \infty$  as  $\text{dist}(z, \phi(\tilde{B})) \rightarrow 0$

■

### Lebesgue-measure on $M$

Given a parametrization  $\phi : B \subset \mathbb{R}^m \rightarrow M$  there exists a  $C^\infty$ -function  $h$  on  $B$  s.t.  $h > 0$  and

$$\int_M f d\mu = \int_B f(\phi(x)) h(x) dx$$

where  $dx$  is Lebesgue-measure on  $\mathbb{R}^m$ . In terms of this function we have the following theorem.

### Theorem 1

Let  $u \in M$  and let  $\phi(B) \rightarrow M$  be a parametrization s.t.  $\phi(0) = u$ . Let  $h$  be the function corresponding to Lebesgue-measure on  $M$ , and let  $f : M \rightarrow \mathbb{C}$  be continuous in a neighbourhood of  $u$ , and bounded everywhere. If  $\frac{m}{2} < p < \frac{m}{2} + \frac{1}{2}$ ,  $z \in \overline{\Omega} \setminus M$  and  $y = y(z)$  is the selected expansion point in the Taylor-approximation, then for  $z$  sufficiently close to  $u$

$$T_p f(z) = Q(z, y)^{\frac{m}{2} - p} \{C(n, p) f(\phi(y)) h(y) + o(1)\}$$

as  $\text{dist}(z, \phi(\tilde{B})) \rightarrow 0$

**Proof**

Put

$$\begin{aligned} H(z, y) &= \int_{\mathbb{R}^m} \frac{f(\phi(y))h(y)dx}{G(z, y, x)^p} = 2^p(f(\phi(y))h(y) \int_{\mathbb{R}^m} \frac{1}{(2G)^p} dx \\ &= Q(z, y)^{m/2-p} C(n, p) f(\phi(y))h(y) \end{aligned}$$

by integration formula B.

It is enough to prove that

$$|H(z, y) - T_p f(z)| \leq |Q(z, y)|^{\frac{m}{2}-p} \cdot o(1)$$

Now choose a ball  $V_4$  and a neighbourhood  $W$  of  $u$  in  $M$  s.t.  $\psi(z - w) = 1$  in  $V_4 \times W$ . Assume  $W \subset \phi(\tilde{B})$  and  $V_4 \cap M \subset W$ . Then

$$|T_p f(z)| \leq \left| \int_w \frac{f(w)d\mu(w)}{F(z, w)^p} \right| + C$$

Since  $|Q(z, y)|^{m/2-p} \rightarrow \infty$  it is enough to prove that

$$|H(z, y) - \int_w \frac{f(w)d\mu(w)}{F(z, w)^p}| \leq |Q(z, y)|^{\frac{m}{2}-p} \cdot o(1).$$

Similarly by choosing a ball  $V_5$  it is enough to estimate

$$|H(z, y) - \int_{|x|<\beta} \frac{f(\phi(x))h(x)dx}{F(z, \phi(x))^p}|$$

where  $\beta > 0$  is chosen so small that  $\phi(\overline{D}(0, \beta)) \subset W$ ,  $f$  is continuous on  $\phi(\overline{D}(0, \beta))$  and  $\beta$  is smaller than the constant  $\alpha_0$  in 1g. Now we have by simple estimation

$$\begin{aligned} &|H(z, y) - \int_{|x|<\beta} \frac{f(\phi(x))h(x)dx}{F(z, \phi(x))^p}| \\ &\leq \left| \int_{|x|<\beta} f(\phi(x))h(x) \left\{ \frac{1}{F^p} - \frac{1}{G^p} \right\} dx \right| \\ &+ \left| \int_{|x|<\beta} \frac{f(\phi(x))h(x) - f(\phi(y))h(y)}{G(z, y, x)^p} dx \right| \\ &+ \left| \int_{|x|\geq\beta} \frac{f(\phi(y))h(y)dx}{G(z, y, x)^p} \right| \end{aligned}$$

Choose a ball  $V_6$  s.t.  $|y(z)| < \frac{\beta}{2}$  for all  $z \in (\bar{\Omega} \setminus M) \cap V_6$ . Then the last term is bounded. By integration lemmas I and II, the first and the second term are bounded by  $|Q(z, y)|^{\frac{m}{2}-p} \cdot o(1)$  and this proves the theorem.

v) ■

### Proposition 2

Given a compact  $K$  in  $M$ , there exist a compact  $\tilde{K}$  in  $M$  with smooth boundary s.t.  $K \subset \text{int } \tilde{K}$ .

### Proof

Since  $M$  is proper we have  $C^3$  change of coordinates. By a refinement of the differentiable structure we can assume that all change of coordinates are  $C^\infty$ . (See e.g. Munkres (8) p. 42). We view  $M$  with this new structure as an abstract  $C^\infty$ -manifold. Then it is easy to find a compactly supported  $C^\infty$  function  $\phi$  with  $\phi \geq 1$  on  $K$ . By Sard's theorem there exists  $c < 1$  a regular value of  $\phi$ . Put  $\tilde{K} = \phi^{-1}([c, \infty))$  a compact with  $C^\infty$ -boundary in the refined structure. In the old structure  $\tilde{K}$  has  $C^3$  boundary, and this proves the proposition.

■

We now define

$$T_p \tilde{K}(z) = \int_{\tilde{K}} \frac{\psi(z-w) d\mu(w)}{F(z, w)^p}$$

We want to prove that  $\text{Re} T_p \tilde{K}(z) \rightarrow \infty$  as we approach  $\tilde{K}$ . We start with a lemma.

### Integration lemma 3

If  $z \in \mathbb{C}$  with  $\text{Re} z > 0$ ,  $a \in \mathbb{R}$  and  $\frac{1}{2} < p < 1$ . Then

$$\text{Re} \int_{\epsilon}^{\infty} (z + 2iat + t^2)^{-p} dt \geq C(p)(\max(\epsilon, 0)^2 + |z + a^2|)^{\frac{1}{2}-p}$$

### Proof

We have  $\text{Re}\{z + 2iat + t^2\} > 0$  for all  $t$ . Then

$$-p\left(\frac{\pi}{2}\right) \leq \arg \{z + 2iat + t^2\}^{-p} \leq p\left(\frac{\pi}{2}\right) \text{ for all } t$$

The same is true when we integrate and

$$\text{Re} \int_{\epsilon}^{\infty} (z + 2iat + t^2)^{-p} dt \geq \cos\left(\frac{p\pi}{2}\right) \left| \int_{\max(\epsilon, 0)}^{\infty} (z + 2iat + t^2)^{-p} dt \right|$$

Without loss of generality we can assume  $\epsilon > 0$ . Assume first  $\epsilon = 1$  and  $\operatorname{Re} z \geq 0$ . Then

$$\begin{aligned} \int_1^\infty (z + 2iat + t^2)^{-p} dt &= |z + a^2|^{\frac{1}{2}-p} \int_0^\infty \{e^{i\theta} + (b+t)^2\}^{-p} dt \\ &\quad - |z + a^2|^{\frac{1}{2}-p} \int_0^{|z+a^2|^{-\frac{1}{2}}} \{e^{i\theta} + (b+t)^2\}^{-p} dt \end{aligned}$$

where  $e^{i\theta} = \frac{z+a^2}{|z+a^2|}$ ,  $b = \frac{ia}{|z+a^2|^{\frac{1}{2}}}$ . Then for some  $\delta > 0$

$$\left| \int_1^\infty (z + 2iat + t^2)^{-p} dt \right| \geq C(p) |z + a^2|^{\frac{1}{2}-p} \text{ when } |z + a^2|^{-\frac{1}{2}} \leq \delta.$$

But  $|z + a^2|^{-\frac{1}{2}} \geq \delta$ ,  $\operatorname{Re} z \geq 0$  defines a compact in  $\mathbb{C} \times \mathbb{R}$  and on this compact obviously  $\left| \int_1^\infty (z + 2iat + t^2)^{-p} dt \right| \geq C(p)$ . This proves the case  $\epsilon = 1$ . The general case is straightforward by a change of variable. ■

### Proposition 3

Let  $[C_{kl}]$  be a strictly positive definite real symmetric  $n \times n$  matrix, and let  $[\gamma_{kl}]$  denote the inverse matrix. If  $z \in \mathbb{C}$  with  $\operatorname{Re} z > 0$ ,  $n/2 < p < \frac{n}{2} + \frac{1}{2}$ ,  $a_1, a_2, \dots, a_n$  are any real numbers and the eigenvalues of  $[C_{kl}]$  are bounded by  $D < \infty$ , then

$$\begin{aligned} I &= \operatorname{Re} \int_{H(\epsilon)} \left\{ z + 2i \sum_{k=1}^n a_k t_k + \sum_{k,l} C_{kl} t_k t_l \right\}^{-p} dt \\ &\geq C(n, p, D) \det[C_{kl}]^{-\frac{1}{2}} \{ \max(\epsilon, 0)^2 + |z + \sum_{k,l} \gamma_{kl} a_k a_l| \}^{\frac{n}{2}-p} \end{aligned}$$

where

$$H(\epsilon) = \{x \in \mathbb{R}^n | x_n \geq \epsilon\}$$

### Proof

We first rotate to diagonalize the quadratic form. By a linear transformation  $S$ ;  $x_k \rightsquigarrow \frac{t_k}{\delta_k}$  ( $\delta_k$  are the eigenvalues) we can assume that all second order coefficients are one.

The Jacobian of  $S$  is  $\det[C_{kl}]^{-\frac{1}{2}}$ .  $S$  does not alter the shape of the domain of integration, but it will move the nearest point to the origin to a distance  $\delta < D|\epsilon|$ . We now rotate back to a position  $H(\delta)$ . Then

$$I = \det\{C_{kl}\}^{-\frac{1}{2}} \operatorname{Re} \int_{H(\delta)} \left\{ z + 2i \sum_{k=1}^n \tilde{a}_k t_k + \sum_{k=1}^n t_k^2 \right\}^{-p} dt$$



where  $\sum_{k=1}^n \tilde{a}_k^2 = \sum_{k,l} \gamma_{kl} a_k a_l$ . We use the integral formula B on the  $n-1$  first coordinates to get

$$I = C(n, p) \det[C_{kl}]^{-\frac{1}{2}} \operatorname{Re} \int_{\delta}^{\infty} \left\{ \left( z + \sum_{k=1}^{n-1} \tilde{a}_k^2 \right) + 2i\tilde{a}_n t_n + t_n^2 \right\}^{n/2-p-\frac{1}{2}} dt_n$$

Integration lemma III applies to this integral, and then

$$\begin{aligned} I &\geq C(n, p) \det[C_{kl}]^{-\frac{1}{2}} \left\{ \max(\delta, 0)^2 + \left| z + \sum_{k=1}^{n-1} \tilde{a}_k^2 + \tilde{a}_n^2 \right| \right\}^{\frac{n}{2}-p} \\ &\geq C(n, p) \det[C_{kl}]^{-\frac{1}{2}} \left\{ D^2 \max(\epsilon, 0)^2 + \left| z + \sum_{k,l} \gamma_{kl} a_k a_l \right| \right\}^{\frac{n}{2}-p} \end{aligned}$$

■

### Theorem 2

If  $m/2 < p < m/2 + \frac{1}{4}$  then

$$\operatorname{Re} T_p \tilde{K}(z) \rightarrow \infty \text{ as } \operatorname{dist}(z, \tilde{K}) \rightarrow 0$$

### Proof

Choose a point  $v \in \tilde{K}$ . Since  $\tilde{K}$  has smooth boundary, there exist a neighbourhood  $W = \phi(B)$  in  $M$  s.t.  $\phi(0) = v$  and  $W \cap \tilde{K} = \phi(B \cap H(\epsilon))$  for some  $\epsilon \leq 0$ . As in theorem 1 it is enough to consider

$$\int_W \frac{\chi_{\tilde{K}}(w) d\mu(w)}{F(z, w)^p} = \int_{B \cap H(\epsilon)} \frac{h(x) dx}{F(z, \phi(x))^p}$$

Put

$$K(z, y) = \int_{H(\epsilon)} \frac{h(y) dx}{G(z, y, x)^p}$$

where  $y = y(z)$  and look at

$$\begin{aligned} |K(z, y) - \int_{B \cap H(\epsilon)} \frac{h(x) dx}{F(z, \phi(y))^p}| &\leq \|h\| \int_{B \cap H(\epsilon)} \frac{|F - G|}{|G|^{p+1}} dx \\ &\quad + \int_{B \cap H(\epsilon)} \frac{|h(x) - h(y)| dx}{|G|^p} \\ &\quad + \int_{H(\epsilon) \setminus B} \frac{|h(y)| dx}{|G|^p} \\ &= I + II + III \end{aligned}$$

If we assume  $|h(x) - h(y)| \leq C|x - y|$  and use 1c

$$II \leq C \int_{B \cap H(\epsilon)} \frac{|x - y| dx}{|G|^p} \leq C \int_{B \cap H(\epsilon)} \frac{|x - y|^3 + |x - y||z - \phi(y)|^2 dx}{|G|^{p+1}}$$

I+II is then exactly the same expression we estimated to get integration lemma 1, and by this lemma

$$\begin{aligned} I + II &\leq C|Q(z, y)|^{m/2-p} \text{dist}(z, \phi(\tilde{B}))^{\frac{1}{2}} + C \\ &\leq C|Q(z, y)|^{\frac{m}{2}-p+\frac{1}{4}} + C \leq C \text{ by } If. \end{aligned}$$

It is easy to see that III is uniformly bounded.

By proposition 3 we have

$$\text{Re}K(z, y) \geq C\{\max\{(\epsilon - y_m), 0\}^2 + |Q(z, y)|\}^{m/2-p} \rightarrow \infty$$

since  $y$  and  $Q(z, y) \rightarrow 0$  as  $z \rightarrow v$

iv)■

### Correction of the $C^\infty$ -kernels

Let  $\epsilon$  be the constant from A, i.e.

$$\text{Re}F(z, w) \geq C|z - w|^2 \text{ if } (z, w) \in \bar{\Omega} \times \partial\Omega \text{ and } |z - w| \leq \epsilon$$

If  $(z, w) \in \bar{\Omega} \times \partial\Omega$  and  $\epsilon/6 \leq |z - w| \leq \epsilon$  then  $\text{Re}F(z, w) \geq C(\Omega, \epsilon)$ . The function  $\text{Re}F(z, w)$  is uniformly continuous on compacts in  $\mathbb{C}^n \times \mathbb{C}^n$  so there exists  $\delta > 0$  s.t.  $\text{Re}F(z, w) \geq C(\Omega, \epsilon)/2$  when  $w \in \partial\Omega$ ,  $\text{dist}(z, \Omega) < \delta$  and  $2\epsilon/3 \geq |z - w| \geq \frac{\epsilon}{5}$ . The cut-off function  $\psi$  was defined s.t.  $\psi(z - w) = 1$  when  $|z - w| < \frac{\epsilon}{4}$  and  $\psi(z - w) = 0$  when  $|z - w| > \frac{\epsilon}{2}$ .

Since  $\Omega$  is strictly pseudoconvex we can find  $\tilde{\Omega}$  pseudoconvex s.t.  $\Omega \subset\subset \tilde{\Omega} \subset\subset \mathbb{C}^n$  and  $\text{dist}(z, \Omega) < \delta$  for all  $z \in \tilde{\Omega}$ . With these definitions it is easy to see that the following (0,1) form  $\nu_w$  is well defined on  $\tilde{\Omega} \setminus (w \in \partial\Omega)$

$$\nu_w(z) = \begin{cases} 0 & \text{when } |z - w| < \frac{\epsilon}{4} \\ \bar{\partial}_z \frac{\psi(z-w)}{F(z,w)^p} & \text{when } 2\epsilon/3 > |z - w| > \frac{\epsilon}{5} \\ 0 & \text{when } |z - w| > \frac{\epsilon}{2} \end{cases}$$

We then have a well defined  $\bar{\partial}$  closed (0,1) form on  $\tilde{\Omega}$  with  $C^\infty$  coefficients, and we can assume that the coefficients are uniformly continuous on  $\tilde{\Omega} \times \partial\Omega$  and that  $\|\nu_w\|_{L^\infty(0,1)} \leq C$  independent of  $w$ .

We now let  $R(z, w)$  be the canonical solution of the  $\bar{\partial}$  problem

$$\bar{\partial}_z R(z, w) = \nu_w.$$

By sup-estimates of the  $\bar{\partial}$ -problem (see e.g. Crantz (9) Corr. 5.2.12 p.186) we have

- a)  $\|R(z, w)\|_{L^\infty(\bar{\Omega})} \leq C$  independent of  $w$ , and
- b)  $\|R(z, \tilde{w}) - R(z, w)\|_{L^\infty(\bar{\Omega})} \leq C\|\nu_{\tilde{w}} - \nu_w\|_{L^\infty(0,1)(\tilde{\Omega})}$

The last relation says that the mapping  $w \mapsto R(z, w)$  is continuous for each  $z \in \bar{\Omega}$ . We then redefine  $T_p f$  and  $T_p \tilde{K}$  by

$$T_p f(z) = \int_M f(w) \left\{ \frac{\psi(z-w)}{F(z, w)^p} - R(z, w) + C \right\} d\mu(w)$$

$$T_p \tilde{K}(z) = \int_{\tilde{K}} \left\{ \frac{\psi(z-w)}{F(z, w)^p} - R(z, w) + C \right\} d\mu(w)$$

where  $C$  is a large constant s.t.

$$Re T_p \tilde{K}(z) > 0 \text{ in } \bar{\Omega} \setminus \tilde{K}.$$

#### End of proof

If we find  $U$  open in  $M$  s.t.  $K \subset U \subset \tilde{K}$  and assume  $m/2 < p < \frac{m}{2} + \frac{1}{4}$  we have an integral extension operator  $T : C_c(U) \rightarrow A(\bar{\Omega})$  defined by

$$Tf(z) = \begin{cases} f(z) & \text{when } z \in U \\ 0 & \text{when } z \in \tilde{K} \setminus U \\ T_p f(z)/T_p \tilde{K}(z) & \text{when } z \in \bar{\Omega} \setminus \tilde{K} \end{cases}$$

The continuity of  $T$  follows from theorem 1 and 2. ■

#### References

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